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Site–bond correlated percolation in a ferromagnetic and antiferromagnetic lattice gas: the Bethe cluster approximation

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Abstract. We study the site–bond correlated percolation problem in a lattice gas within the Bethe cluster approximation and obtain explicit results both in the ferromagnetic and antiferromagnetic regions.

1. Introduction

The site–bond correlated percolation (SBCP) problem, which recently has been studied by many authors, is a generalisation of the correlated percolation problem, i.e. the clusters in a lattice gas or Ising model are defined as the maximal sets of nearest-neighbour particles connected by active bonds. The probability of a bond being active is p_B and non-active $1-p_B$. For ferromagnetic interactions this formulation has been proven suitable to describe the effect of a bad solvent in gelation (Coniglio *et al* 1979, Coniglio *et al* 1982b), and also to describe the droplets in a lattice gas or Ising model (Coniglio and Klein 1980). A Monte Carlo study of the SBCP in two and three dimensions has also been done (Stauffer 1981, Heermann and Stauffer 1981, Roussenq 1981). On the other hand, very little attention has been devoted to the antiferromagnetic case.

Some authors have studied the site correlated percolation problem with antiferromagnetic interaction (Müller-Krumbhaar 1974, Stoll and Domb 1979, Murata 1979, Napiorkowski and Hemmer 1980). More recently, the antiferromagnetic lattice gas has also been investigated in the context of the SBCP (Coniglio *et al* 1981) in $d=2$ with the Migdal–Kadanoff renormalisation group.

In this paper we consider the SBCP problem for both ferromagnetic and antiferromagnetic interactions in the Bethe cluster approximation. We extend the study of the ferromagnetic case (Coniglio *et al* 1979, 1982b) to include the coexistence region; in this region we investigate the behaviour of the 'droplets' (i.e. of the clusters with $p_B = 1-e^{-K/2}$); we find that these droplets do not diverge along the spinodal line (where the susceptibility diverges). Therefore at least in the Bethe approximation these droplets do not seem suitable to describe the spinodal line.

In the antiferromagnetic region we find that the critical percolation lines for all p_B end at the top of the phase boundary at $T=0$. For some p_B they intersect the phase boundary in a point which therefore is critical both for percolation and for the

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staggered susceptibility. This behaviour is not found at $d = 2$ (Coniglio *et al* 1981), and therefore could be typical of systems of higher dimensionality. This antiferromagnetic model could be of relevance in the study of the sol–gel transitions in a good solvent.

In § 2 we give a review of the site–bond correlated percolation problem as the $Q = 1$ limit of a lattice gas Potts model. In § 3 we apply the Bethe cluster approximation to both the ferromagnetic and antiferromagnetic case and draw the conclusions.

2. Site–bond correlated percolation. The Q-state lattice gas Potts model

It is well known (Kasteleyn and Fortuin 1969) that the random bond percolation problem can be obtained from the Q-state Potts model in the $Q = 1$ limit. Analogously, it is well established (Wu 1978, Murata 1979, Coniglio and Klein 1980, Coniglio *et al* 1981) that the lattice gas Potts model in the $Q = 1$ limit gives rise to the site–bond correlated percolation. In the site–bond correlated percolation problem the sites are correlated as in a lattice gas, but the bonds between neighbouring sites are active with probability p_B and non-active with probability $1 - p_B$.

Denote

$$-\beta\mathcal{H}_{LG} = K \sum_{\langle ij \rangle} n_i n_j + \Delta \sum_i n_i, \tag{1}$$

the lattice gas Hamiltonian on a regular lattice of N sites where $n = 1$ if site i is occupied, 0 otherwise, $\beta = 1/K_B T$, K is the nearest-neighbour coupling constant related to the Ising coupling constant K_I by $K_I = K/4$, Δ is the chemical potential related to the Ising magnetic field H and the coordination number $z = \sigma + 1$ by $-H = \frac{1}{2}(\Delta + \frac{1}{2}zK)$. Positive and negative values of K correspond respectively to ferromagnetic and antiferromagnetic coupling. The sum $\Sigma_{\langle ij \rangle}$ is over nearest neighbours.

In the site–bond correlated percolation the averages of the quantities of interest are calculated as follows:

$$\langle \ldots \rangle = \lim_{N \to \infty} \sum_{\{n_i\}} \exp(-\beta\mathcal{H}_{LG}) \Big(\sum_{C \subseteq E\{n_i\}} p_B^{|C|} (1 - p_B)^{|D|} \Big) \Big(\sum_{\{n_i\}} \exp -\beta\mathcal{H}_{LG} \Big)^{-1} \tag{2}$$

where $E\{n_i\}$ is the set of all bonds in the sublattice made of the occupied sites in the configuration $\{n_i\}$, C is a subset of $E\{n_i\}$ and $D = E\{n_i\} - C$. $|C|$ and $|D|$ are the number of bonds respectively in the subsets C and D. Moreover, for every configuration $\{n_i\}$ we have $\Sigma_{C \subseteq E\{n_i\}} p_B^{|C|} (1 - p_B^{|D|}) = 1$.

The same probability distribution as in (2) can be obtained, as said before, from the Hamiltonian

$$-\beta\mathcal{H} = -\beta\mathcal{H}_{LG} - \beta\mathcal{H}_{DP} \tag{3}$$

in the $Q = 1$ limit, where $-\beta\mathcal{H}_{DP}$ is the diluted Potts Hamiltonian

$$-\beta\mathcal{H}_{DP} = J \sum_{\langle ij \rangle} (\delta_{\sigma_i \sigma_j} - 1) n_i n_j + h \sum_i (\delta_{\sigma_i 1} - 1) n_i \tag{4}$$

and σ_i is the Q-state Potts variable. This procedure is described in detail e.g. in Coniglio *et al* (1981), and is based essentially on the fact that the partition function for (3) is written as

$$Z = \sum_{\{n_i\}} \exp(-\beta\mathcal{H}_{LG}) Q^{\Sigma_i (1 - n_i)} Z_{\text{Potts}}\{n_i\} \tag{5}$$

with

$$Z_{\text{Potts}}\{n_i\} = \sum_{C \subseteq E\{n_i\}} p^{|C|} q^{|D|} \prod_r [(Q-1)\exp(-hs_r)+1] \tag{6}$$

where $q = e^{-J}$, $p = 1-q$, r labels the clusters in the configuration C, s_r is the number of sites in the rth cluster and the product is over all the clusters in C.

By means of (5) and (6) one can see that in the $Q = 1$ limit, the same probability distribution as in (2) is obtained, provided that $p_B = 1 - e^{-J} = p$. In § 3 we shall make use of the ghost field h description (Kasteleyn and Fortuin 1969, Reynolds *et al* 1977) to derive the expressions for the percolative functions of interest, that is:

the density of particles

$$\rho = \left\langle \sum_i n_i \right\rangle_{h=0} \Big/ N; \tag{7}$$

the percolation probability

$$\mathbb{P} = \lim_{h \to 0} \mathbb{P}_h \tag{8}$$

where

$$\mathbb{P}_h = 1 - \sum{}' s\langle n_s \rangle / \rho, \tag{9}$$

the sum Σ' is over all finite clusters and the quantity $\Sigma' s\langle n_s\rangle/\rho$ in the $h = 0$ limit is the probability that a site belongs to a finite cluster;

the mean cluster size

$$S = \lim_{h \to 0} \sum{}' s^2 \langle n_s \rangle / \sum s\langle n_s \rangle \tag{10}$$

which can be written by means of (9)

$$S = \left(\frac{1}{1-\mathbb{P}_h} \frac{d\mathbb{P}_h}{dh} \right)_{h=0}. \tag{11}$$

In the above Hamiltonian formalism, quantities (7) and (9) can be calculated as follows (Murata 1979):

$$\rho = \lim_{\substack{N \to \infty \\ Q \to 1 \\ h \to 0}} \frac{1}{N} \frac{\partial}{\partial \Delta} \ln Z \tag{12}$$

$$\sum{}' s\langle n_s \rangle \equiv \rho_f = \lim_{\substack{N \to \infty \\ Q \to 1}} \frac{1}{N} \frac{1}{1-Q} \frac{\partial}{\partial h} \ln Z. \tag{13}$$

By means of (12) and (13) the percolation probability can be easily calculated:

$$\mathbb{P} = \lim_{h \to 0} (1 - \rho_f/\rho). \tag{14}$$

3. Percolation threshold in the Bethe cluster approximation

For the Ising model on a Bethe lattice it is known that at $H = 0$ the percolation critical point does not coincide with the Bethe critical point $1/K_c = -2/\ln[1-2/(\sigma+1)]$

(Coniglio 1975, Essam 1980). In particular, because of the large connectivity of the Bethe lattice, the infinite cluster appears below the critical point as happens in dimensions $d \geqslant 3$ (Müller-Krumbhaar 1974). In the terminology introduced in § 2, the cluster made of nearest-neighbour particles, which is generally called the 'Ising cluster', is obtained in the $Q = 1$ limit if one sets $p_B = 1$. As has been pointed out in Coniglio and Klein (1980) and is explicitly given in Coniglio *et al* (1981), the Hamiltonian (3) for $h = 0$ is equivalent to the following symmetric $(Q+1)$-state Potts model,

$$-\beta\mathcal{H}(b_i) = J\sum_{\langle ij\rangle}(\delta_{b_i b_j}-1) - 2(J-\tfrac{1}{2}K)\sum_{ij}\delta_{b_i 0}\delta_{b_j 0} + [2H + z(J-\tfrac{1}{2}K) + \ln Q]\sum_i \delta_{b_i 0} \qquad (15)$$

where $-H = \frac{1}{2}(\Delta + \frac{1}{2}zK)$ and b_i is a $(Q+1)$-valued variable related to the (σ, n) variables by the transformation

$$b_i = \begin{cases} 1 \ldots Q & \text{if } n_i = 1 \text{ and } \sigma_i = 1 \ldots Q \\ 0 & \text{if } n_i = 0. \end{cases}$$

Hamiltonian (15) in the $Q = 1$ limit for $J = K/2$ is a symmetric two-state Potts model with coupling constant $K/2$, i.e. a lattice gas model with coupling constant K.

If we call (Coniglio and Klein 1980) 'Ising droplets' the clusters made of nearest-neighbour particles connected by active bonds with probability $p_B = 1 - e^{-K/2}$, by the above argument we expect that the Ising critical point $K = K_c$, $H = 0$ is also a percolation point and that the clusters diverge at this point with Ising exponents. This argument is given in Coniglio and Klein (1980) and is tested therein with the Migdal–Kadanoff renormalisation group (MKRG) on a triangular lattice. On the square lattice it has also been tested by means of the MKRG, for ferromagnetic interactions (Coniglio *et al* 1981), while it is evident that for antiferromagnetic interactions ($K < 0$) the asymmetric term in Hamiltonian (15) can never vanish.

In this section we study the Hamiltonian (3) in the Bethe cluster approximation (Domb 1960), both in the ferromagnetic and antiferromagnetic case, following the formalism of Murata (1979) but always keeping J finite.

For this purpose let us consider an elementary cluster of a central site and its $\sigma + 1$ nearest neighbours (figure 1); let Δ and h be respectively the chemical potential and the ghost field acting on the central site. The influence of the rest of the lattice is taken into account by means of 'auxiliary fields' Δ' and h' acting on the near neighbours. The partition function on this elementary cluster is

$$Z_c = Q[Q + e^{\Delta'} + (Q-1)e^{\Delta'-h'}]^{\sigma+1} + e^{\Delta}[Q + e^{K+\Delta'} + (Q-1)e^{K+\Delta'-h'}]^{\sigma+1}$$
$$+ (Q-1)e^{\Delta-h}[Q + e^{K+\Delta'}q + e^{K+\Delta'-h'} + (Q-2)e^{K+\Delta'-h'}q]^{\sigma+1} \qquad (16)$$

where $q = e^{-J}$.

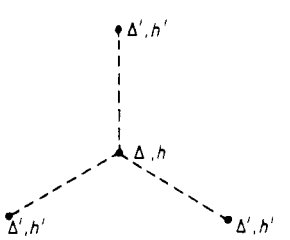

Figure 1. Reduced lattice for $\sigma = 2$. Δ is the chemical potential, h the ghost field. Δ' and h' are the 'auxiliary' fields.

The fields Δ' and h' are determined by requiring the translational invariance conditions

$$\rho = \lim_{\substack{Q\to 1\\ h\to 0}} \frac{\mathrm{d}}{\mathrm{d}\Delta} \ln Z_c = \lim_{\substack{Q\to 1\\ h\to 0}} \frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}\Delta'} \ln Z_c$$

and

$$\rho_f = \lim_{Q\to 1} \frac{1}{(1-Q)}\frac{\mathrm{d}}{\mathrm{d}h} \ln Z_c = \lim_{Q\to 1} \frac{1}{z}\frac{1}{(1-Q)}\frac{\mathrm{d}}{\mathrm{d}h'} \ln Z_c$$

which gives respectively the following equations for Δ' and h':

$$\mathrm{e}^{\Delta} = \mathrm{e}^{\Delta'}(1+\mathrm{e}^{\Delta'})^{\sigma}/(1+\mathrm{e}^{\Delta'+K})^{\sigma} \tag{17}$$

and

$$\mathrm{e}^{\Delta-h-\Delta'+h'} = (1+\mathrm{e}^{\Delta'})^{\sigma}/[1+q\,\mathrm{e}^{K+\Delta'}+(1-q)\,\mathrm{e}^{K+\Delta'-h'}]^{\sigma}. \tag{18}$$

The density

$$\rho = \mathrm{e}^{\Delta}(1+\mathrm{e}^{\Delta'+K})^{\sigma+1}/[(\mathrm{e}^{\Delta'}+1)^{\sigma+1}+\mathrm{e}^{\Delta}(1+\mathrm{e}^{\Delta'+K})^{\sigma+1}] \tag{19}$$

can be written by means of (17)

$$\rho = \mathrm{e}^{\Delta'}(1+\mathrm{e}^{\Delta'+K})/[(1+\mathrm{e}^{\Delta'})+\mathrm{e}^{\Delta'}(1+\mathrm{e}^{\Delta'+K})] \tag{20}$$

which gives for the auxiliary field Δ'

$$\mathrm{e}^{-\Delta'} = \{(1-2\rho)+[(1-2\rho)^2+4\mathrm{e}^{K}\rho(1-\rho)]^{1/2}\}/2\rho. \tag{21}$$

In order to write an expression for the percolation probability (8) let us evaluate

$$\mathbb{P}_h = 1-\rho_f/\rho. \tag{22}$$

In fact, we can easily evaluate ρ_f/ρ

$$\rho_f/\rho = \mathrm{e}^{\Delta-h}\frac{[1+q\,\mathrm{e}^{\Delta'+K}+(1-q)\,\mathrm{e}^{\Delta'+K-h'}]^{\sigma+1}}{\mathrm{e}^{\Delta}(1+\mathrm{e}^{\Delta'+K})^{\sigma+1}} \equiv \mathrm{e}^{-h}\mathbb{Q}^{\sigma+1} \tag{23}$$

where

$$\mathbb{Q} = [1+q\,\mathrm{e}^{\Delta'+K}+(1-q)\,\mathrm{e}^{\Delta'+K-h'}]/(1+\mathrm{e}^{\Delta'+K}). \tag{24}$$

For non-zero ghost field h, equation (23) can be written as

$$\rho_f/\rho = (1-p_h)\mathbb{Q}^{\sigma+1}$$

where

$$p_h = 1-\mathrm{e}^{-h}, \tag{25}$$

and equation (22) as

$$\mathbb{P}_h = 1-(1-p_h)\mathbb{Q}^{\sigma+1}. \tag{26}$$

Relation (26) has been derived (Coniglio *et al* 1982b) in a more general context. This equation can be understood as introducing a 'ghost site' (Kasteleyn and Fortuin 1969) (figure 2). We add to the original lattice a ghost site such that every site in the original lattice is connected to it with probability p_h. This implies that for $h \neq 0$ we can have an infinite cluster made of all the sites that are connected through the ghost site.

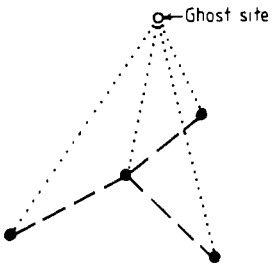

Figure 2. The open circle is the ghost site; the dotted lines are ghost bonds.

Therefore, in equation (26), $(1-p_h)\mathbb{Q}^{\sigma+1}$ can be regarded as the probability that a site is disconnected from the ghost site and that there is no infinite cluster in any of the $(\sigma+1)$ directions that move away from that site. Of course $p_h = 1-e^{-h}$ is the probability that a site is connected with the ghost site (Sykes and Gaunt 1976, Reynolds *et al* 1977). From equation (24) one can find a recurrence equation for $\mathbb{Q}$.

Define

$$1-\hat{\rho} = (1+q\ e^{K+\Delta'})/(1+e^{K+\Delta'}),$$

that is

$$\hat{\rho} = (1-q)\ e^{\Delta'+K}/(1+e^{\Delta'-K}) = p_B\ e^{\Delta'+K}/(1+e^{\Delta'+K}); \tag{27}$$

therefore

$$\mathbb{Q} = 1-\hat{\rho}+\hat{\rho}\ e^{-h'}.$$

Using equations (17) and (18), one can show that

$$e^{-h'} = e^{-h}\mathbb{Q}^{\sigma} = (1-p_h)\mathbb{Q}^{\sigma},$$

and therefore we have the following recursion relation for $\mathbb{Q}$:

$$\mathbb{Q} = (1-\hat{\rho})+(1-p_h)\hat{\rho}\mathbb{Q}^{\sigma} \tag{28}$$

which in the $h=0$ limit is the same equation as that studied by Fisher and Essam (1961) for the random site percolation.

To find the percolation threshold let us calculate the mean cluster size

$$S = [(1-\mathbb{P}_h)^{-1}\,d\mathbb{P}_h/dh]_{h=0};$$

by means of (26) and (28) we obtain

$$S = (1+\hat{\rho}\mathbb{Q}^{\sigma-1})/(1-\sigma\hat{\rho}\mathbb{Q}^{\sigma-1}). \tag{29}$$

The percolative threshold is therefore given (Fisher and Essam 1961) by

$$\hat{\rho}_{crit} = 1/\sigma. \tag{30}$$

This equation, by means of the definition of $\hat{\rho}$, equation (27), gives the critical value of the auxiliary field Δ', which replaced in (20) allows us to write the critical value of the density for the percolation

$$\rho_{crit} = p_B\sigma/[(p_B\sigma-1)^2\ e^{K}+2p_B\sigma-1]. \tag{31}$$

This equation can also be given in the H and K variables by means of equations (17)

and (19), that is

$$e^{2H} = \{p_B\sigma\, e^{K/2}/[(p_B\sigma - 1)\, e^{K} + 1]\}^{\sigma}\, e^{K/2}(p_B\sigma - 1). \tag{32}$$

Equation (31) has been obtained by Coniglio *et al* (1979) for Cayley tree types of lattices and for $p_B = 1$ gives the same result as Murata (1979) and Kikuchi (1970).

Let us study equation (32) separately in the ferromagnetic and antiferromagnetic cases.

3.1. Ferromagnetic case

The critical percolation line (32) has been studied in the $K > 0$ region and compared for $T < T_c$ with the spinodal line where the susceptibility diverges given by

$$e^{2H} = \left(\frac{(\sigma+1)\, e^{-K} + (\sigma-1) \pm g}{2\sigma\, e^{-K/2}}\right)^{\sigma} \left(\frac{(\sigma+1)\, e^{-K} - (\sigma-1) \pm g}{2e^{-K/2}}\right) \tag{33}$$

where

$$g^2 = (1 - e^{-K})[(\sigma-1)^2 - (\sigma+1)^2\, e^{-K}].$$

Equation (32) shows that the critical Ising point $H = 0$, $K = K_c$ is also a percolation point, with p_B given by

$$p_B = 1 - e^{-K_c/2} = 2/(\sigma+1)$$

as we expected from the argument given at the beginning of this section. In figure 3 we have plotted in the (H, T) plane the spinodal line together with the critical line of percolation points for $p_B = 1 - e^{-K/2}$ and $p_B = 1$ for $\sigma = 11$. In figure 4 the same

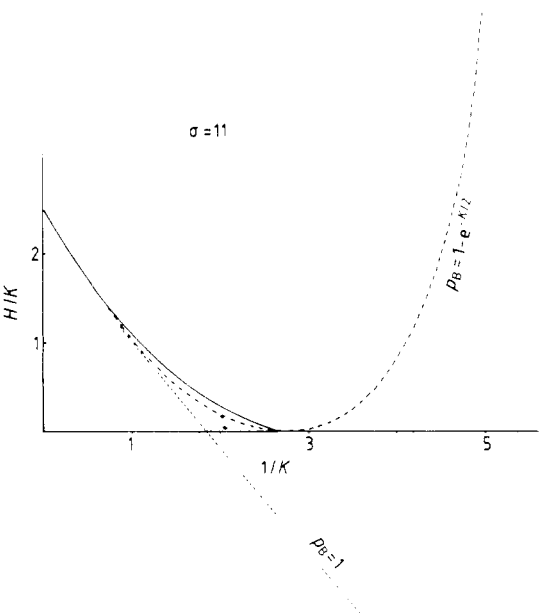

Figure 3. The percolation line for $\sigma = 11$. $p_B = 1 - e^{-K/2}$ (broken line) and $p_B = 1$ (dotted line) in the (H, T) plane. The full line is the spinodal.

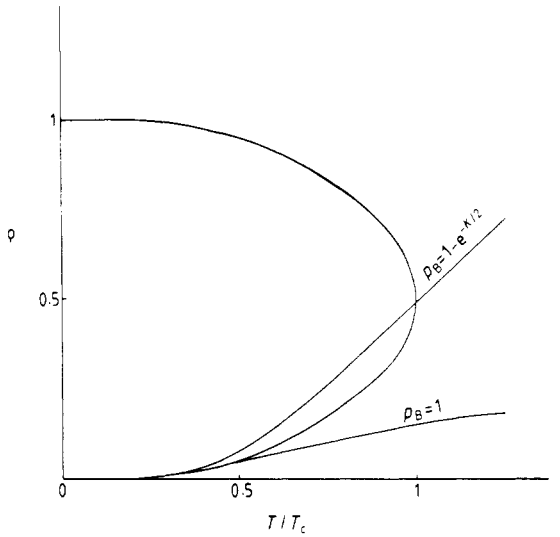

Figure 4. The spinodal and the percolation lines for $p_B = 1$ and $p_B = 1 - e^{-K/2}$ for $\sigma = 3$ in the $(\rho, T/T_c)$ plane.

curves are shown for $\sigma = 3$ in the $(\rho, T/T_c)$ plane. The results shown in the above figures are in agreement with Coniglio *et al* (1979, 1982b) who restricted their analysis to the region outside the coexistence curve. Here we have investigated the behaviour of the droplets with $p_B = 1 - e^{-K/2}$ also in the coexistence region: as one can see, these droplets do not diverge along the spinodal line. Therefore, at least in the Bethe approximation, a new definition of clusters seems necessary for $H \neq 0$ to describe the droplets appropriately. Analogous indications come from considering the critical behaviour. The critical exponent γ for the zero magnetic field susceptibility does not coincide with the value of the critical exponent γ'_p for the mean cluster size (Coniglio *et al* 1982b), while it is known that above T_c they coincide.

3.2. Antiferromagnetic case

In the antiferromagnetic region one must consider two sublattices (figure 5) of sites A and B. On these two sublattices we define the partition functions

$$Z_A = Z_C(\Delta' \to \Delta_A, h' \to h_A) \qquad Z_B = Z_C(\Delta' \to \Delta_B, h' \to h_B).$$

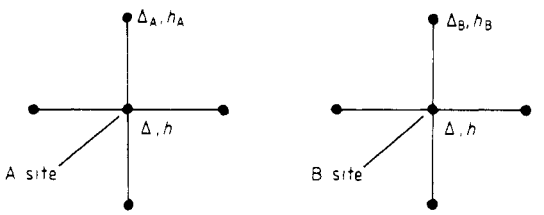

Figure 5. Two reduced lattices for $\sigma = 3$ centred respectively on site A and B. Δ_A, h_A are the auxiliary fields acting on the neighbours of A, while Δ_B, h_B act on the neighbours of B.

The transitional invariance requirements are now

$$\rho_A = \rho'_B \qquad \rho'_A = \rho_B \tag{34}$$

where e.g.

$$\rho_A = \lim_{\substack{Q\to 1\\ h\to 0}} \frac{\partial}{\partial \Delta} \ln Z_A \qquad \rho'_B = \lim_{\substack{Q\to 1\\ h\to 0}} \frac{1}{Z}\frac{\partial}{\partial \Delta_B} \ln Z_B$$

and

$$\rho_{fA} = \rho'_{fB} \qquad \rho'_{fA} = \rho_{fB} \tag{35}$$

where

$$\rho_{fA} = \lim_{Q\to 1} \frac{1}{1-Q}\frac{\partial}{\partial h} \ln Z_A \qquad \rho'_{fB} = \lim_{Q\to 1} \frac{1}{1-Q}\frac{1}{Z}\frac{\partial}{\partial h_B} \ln Z_B.$$

In order to find the critical line of percolation points, one can define the probability of an infinite cluster with origin in A and the probability of an infinite cluster with origin in B respectively as

$$\mathbb{P}_A = \lim_{h\to 0} \mathbb{P}_h(A) = \lim_{h\to 0} (1-\rho_{fA}/\rho_A) \qquad \mathbb{P}_B = \lim_{h\to 0} \mathbb{P}_h(B) = \lim_{h\to 0} (1-\rho_{fB}/\rho_B).$$

Using equations (34) and (35), $\mathbb{P}_h(A)$ and $\mathbb{P}_h(B)$ can be written as

$$\mathbb{P}_h(A) = 1-(1-p_h)\mathbb{Q}_A^{\sigma+1} \qquad \mathbb{P}_h(B) = 1-(1-p_h)\mathbb{Q}_B^{\sigma+1}$$

where

$$\mathbb{Q}_A = \mathbb{Q}(\Delta' \to \Delta A,\ h' - h_A) \qquad \mathbb{Q}_B = Q(\Delta' \to \Delta_B,\ h' \to h_B) \tag{36}$$

If one defines the quantities

$$\hat{\rho}_A = p_B\, \mathrm{e}^{\Delta_A + K}/(1+\mathrm{e}^{\Delta_A+K})$$

and

$$\hat{\rho}_B = p_B\, \mathrm{e}^{\Delta_B + K}/(1+\mathrm{e}^{\Delta_B+K})$$

we have the following recursion relation for (36):

$$\mathbb{Q}_A = (1-\hat{\rho}_A)+(1-p_h)\hat{\rho}_A\mathbb{Q}_B^{\sigma} \qquad \mathbb{Q}_B = (1-\hat{\rho}_B)+(1-p_h)\hat{\rho}_B\mathbb{Q}_A^{\sigma}. \tag{37}$$

By means of (37) the percolation thresholds for $\mathbb{P}_A$ and $\mathbb{P}_B$ are given by

$$\hat{\rho}_A\hat{\rho}_B = 1/\sigma^2. \tag{38}$$

Equations (34) and (38) can be solved numerically. The results are plotted in figures 6 and 7. In figure 6 we have plotted in the $(|H|/|K|,\ 1/|K|)$ plane the antiferromagnetic phase boundary given by (Kasteleyn 1956, di Liberto *et al* 1982)

$$\mathrm{e}^{2H} = \left(\frac{(\sigma-1)\,\mathrm{e}^{-K}+(\sigma+1)\pm m}{2\sigma\,\mathrm{e}^{-K/2}}\right)^{\sigma}\left(\frac{(\sigma-1)\,\mathrm{e}^{-K}-(\sigma+1)\pm m}{2\,\mathrm{e}^{-K/2}}\right)$$

where

$$m^2 = (1-\mathrm{e}^{-K})[(\sigma+1)^2-(\sigma-1)^2\,\mathrm{e}^{-K}]$$

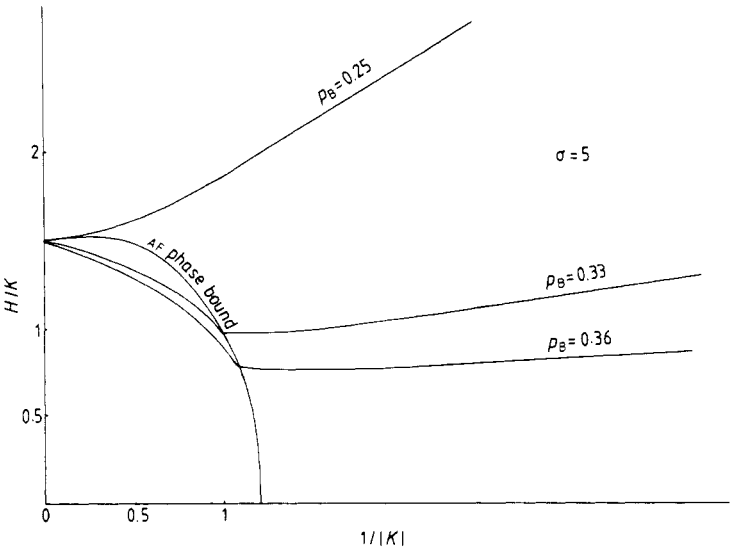

Figure 6. The antiferromagnetic phase boundary and the percolation lines for $p_B = 0.25$, $p_B = 0.33$ and $p_B = 0.36$ for $\sigma = 5$ in the $(|H|, T)$ plane.

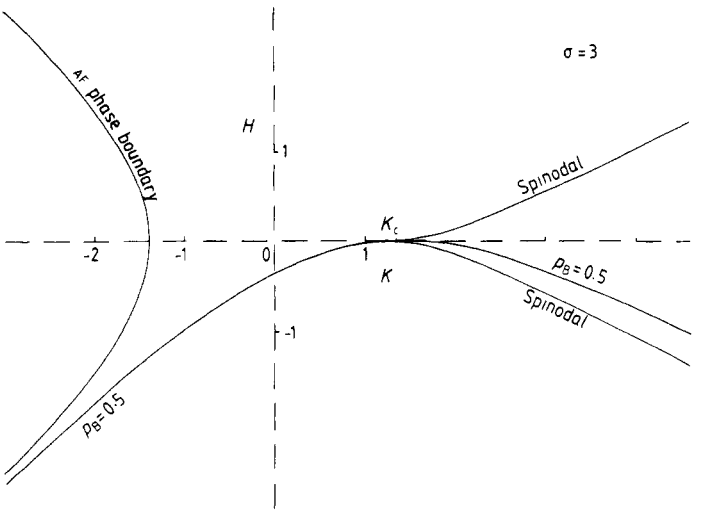

Figure 7. An overall picture of percolation line for $p_B = 0.5$ in the ferromagnetic and antiferromagnetic region together with the spinodal and the antiferromagnetic phase boundary in the (H, K) plane for $\sigma = 3$.

and the critical line of percolation points for $\sigma = 5$ and $p_B = 0.25$, $p_B = 0.33$ and $p_B = 0.36$.

It is significant to remark that at $T = 0$ and $H/K \leq (\sigma + 1)/4$ the ground state is ordered antiferromagnetically, and therefore we expect no percolation there and as a matter of fact all the percolation lines end at the top of the phase boundary. Smaller values of p_B determine a larger percolative region. Moreover, we find that for most p_B the percolation line intersects the antiferromagnetic phase boundary. Therefore

at this point we have both percolation and divergence of the staggered susceptibility: an interesting question would be to investigate the behaviour of the critical indices at this intersection point. The same behaviour could probably be found in three-dimensional systems.

The case studied here (Bethe lattice) gives the same qualitative picture as for the two-dimensional case (Coniglio *et al* 1981) only at $T = 0$. An overall picture of the percolation line in the ferromagnetic and antiferromagnetic regions is shown in figure 7 for $p_B = 0.5$, $\sigma = 3$ together with the spinodal and the antiferromagnetic phase boundary in the (H, K) plane.

In the ferromagnetic case for $p_B = 1 - e^{-K_c/2}$ the clusters diverge at the Ising critical point; an analogous behaviour does not occur in the antiferromagnetic case, i.e. the droplets with $p_B = 1 - e^{-K/2}$ do not diverge along the antiferromagnetic phase boundary as expected. A new definition of clusters of holes and particles has been recently introduced (Coniglio *et al* 1982a) which is a better candidate to describe the antiferromagnetic droplets. The study of this model is under investigation.

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